Exploring Topological Data Analysis for Identifying Phenomenological Stochastic Bifurcations

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Abstract

Changes in the parameters of dynamical systems can cause the state of the system to shift between different qualitative regimes, called bifurcations. In stochastic dynamical systems, particularly P-type bifurcations are difficult to quantify. Currently, the common practice is to visually analyze the probability density function to determine the type of state, but this approach is limited to experienced users, systems with small state spaces and mandate human intervention. In contrast, this study presents a new approach based on Topological Data Analysis (TDA) that uses the superlevel persistence of the probability or kernel density function to mathematically quantify P-type bifurcations.

Motivation

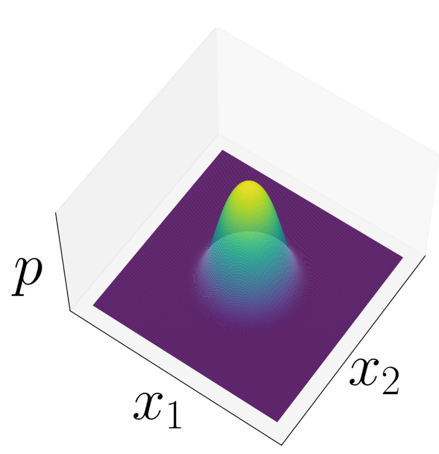
Determination of the state of a stochastic dynamical system and the inception of a bifurcation is of immense interest to accurately understand, predict and control the behaviour of systems, e.g.:

- Population Growth
- Aeroelastic Systems
- Financial Markets

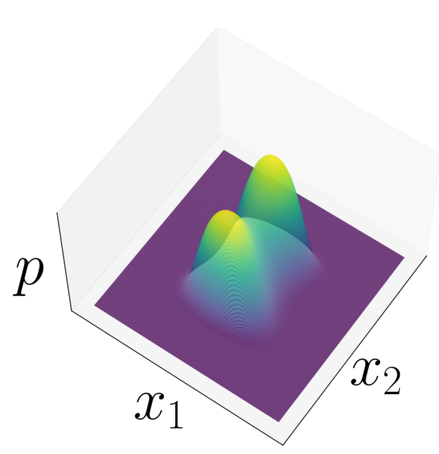


P-Bifurcations

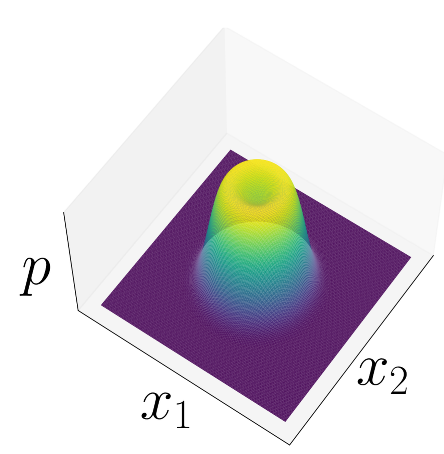
Phenomenological (P-type) bifurcations are characterized by topological changes in the **probability density function**. The changes can be shifts from monostability to bistability or limit cycle oscillations or to more complex shapes:



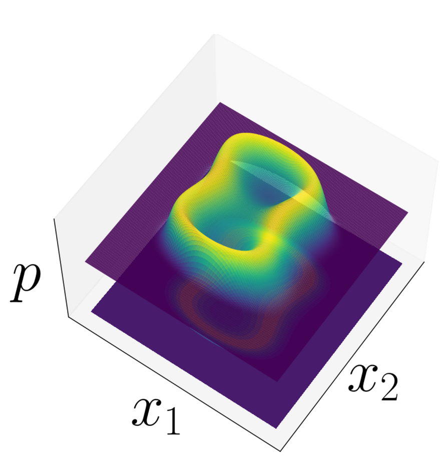
Monostable



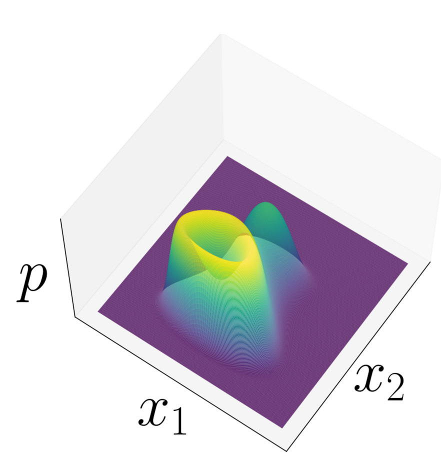
Bistable



Limit Cycle



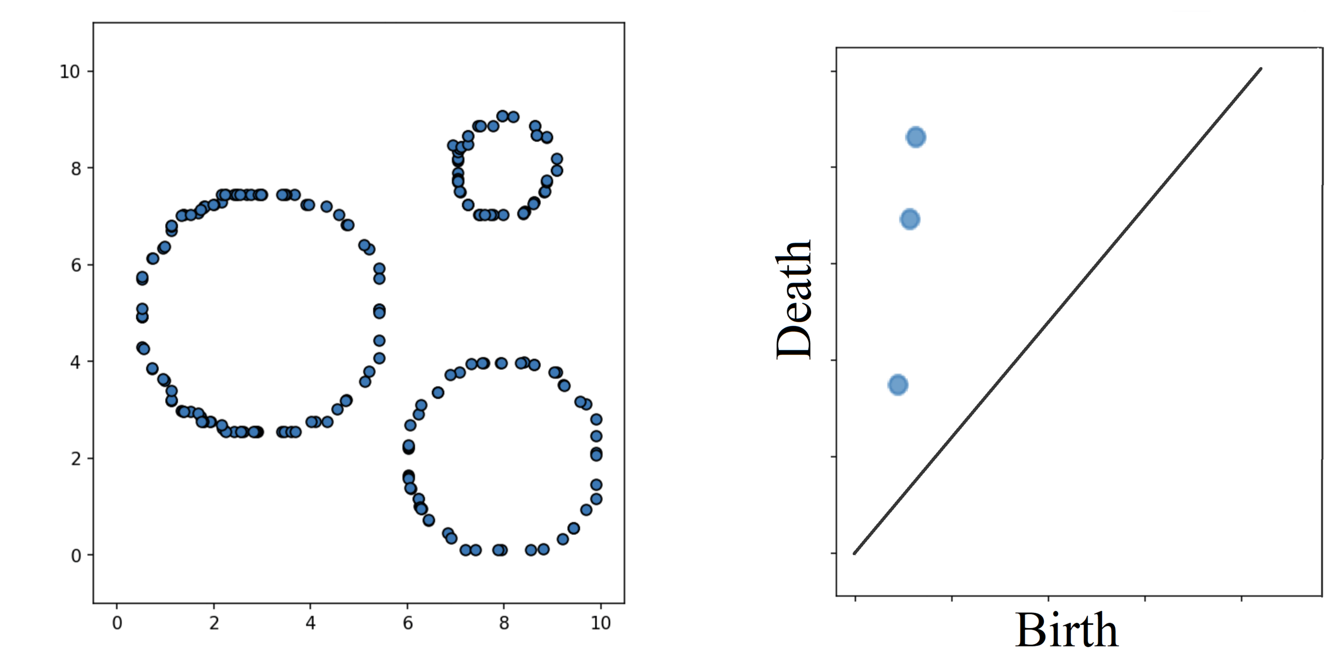
Inverted Bistable



Monostable with Limit Cycle

Point Cloud Persistent Homology

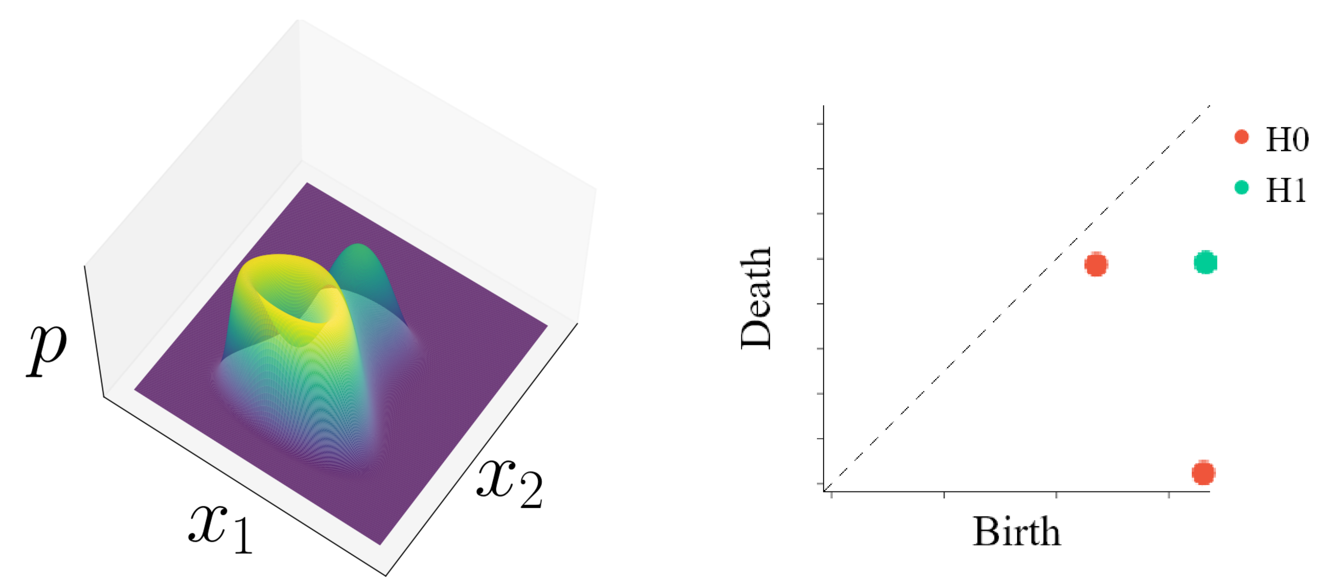
Persistent homology captures information about the shape of a parameterized space by tracking how its homology changes as the parameter varies. For point clouds, that parameter is a radius r of balls centred around each point.



1. Take a point cloud $\chi =_1, \cdots, x_n\} \subset^d$ given as input.
2. For a fixed $r \geq 0$, the Rips complex is given by
$$R(\chi, r) = \{\sigma = \{x_0, \cdots, x_d\} \mid \|x_i - x_j\| \leq r \text{ for all } i, j\}.$$
3. Raise the r from $0 \to \infty$, and construct a sublevelset filtration, along with a persistence diagram accordingly.

Cubical Persistent Homology

Persistent homology can also be computed for image data. For such data, the filtration is based on the height of the data points, and persistence diagram shows the number of connected components and loops (H_0 and H_1 classes respectively).



Method

Given a Probability Density

1. Compute cubical persistence for different bifurcation parameter values.
2. Quantify a bifurcation using a change in ranks of homology groups.

Given a Kernel Density

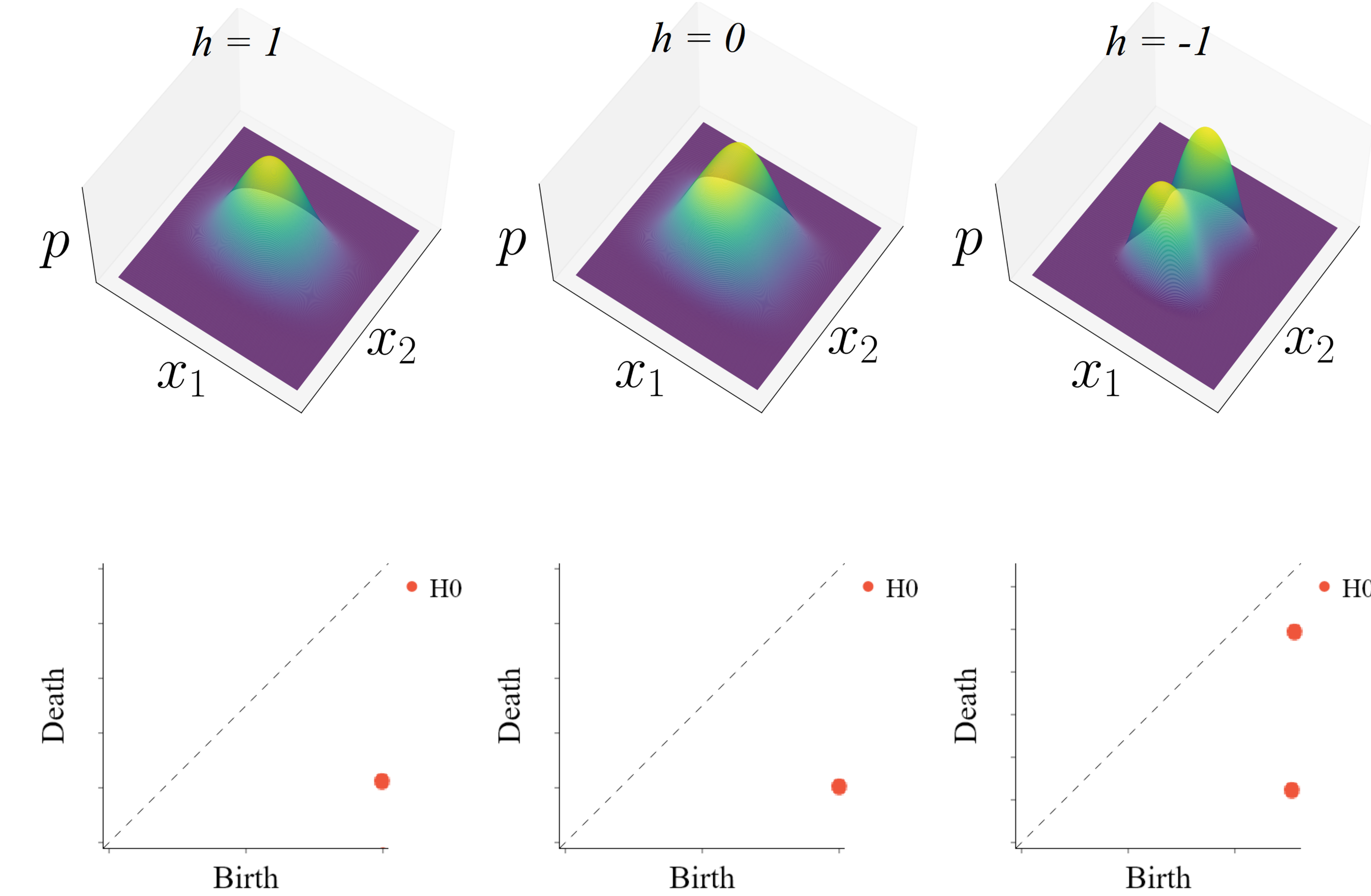
1. Compute point cloud persistent homology for different bifurcation parameter values at various height levels. For topological consistency using noisy kernel density, see [1].
2. Observe changes in ranks of homology groups.

Example: Duffing Oscillator [2]

$$\ddot{X} + \dot{X} + hX + X^3 = dW$$

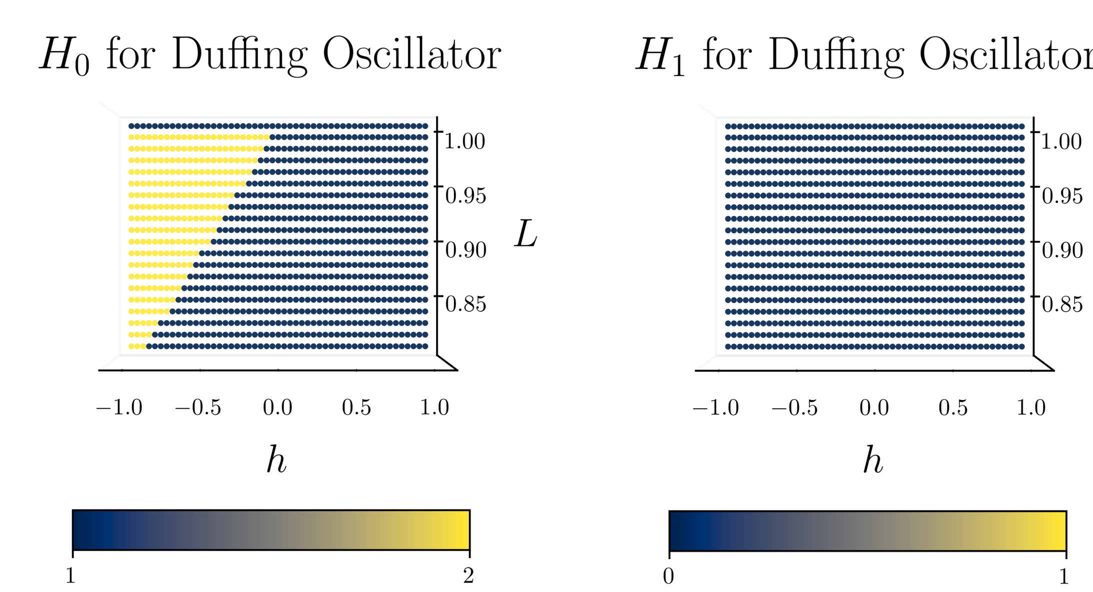
$$p_{x_1 x_2}(\mathbf{x}) = C \exp\left[-\frac{1}{2}\left(x_2^2 + hx_1^2 + \frac{1}{2}x_1^4\right)\right]$$

Probability Density



Rank of H_0 class changes from 1 to 2 with no change in H_1

Kernel Density



Rank of H_0 class changes from 1 to 2 with no change in H_1

Conclusion

1. A novel tool for detecting P-bifurcations and knowing the state of the system.
2. A change in the topology of the PDF or KDE results in an abrupt change in the ranks of various homology classes.

References

[1] Omer Bobrowski, Sayan Mukherjee, and Jonathan E. Taylor. Topological consistency via kernel estimation. *Bernoulli*, 23(1), February 2017.

[2] K.I. Mamis and G.A. Athanassoulis. Exact stationary solutions to fokker–planck–kolmogorov equation for oscillators using a new splitting technique and a new class of stochastically equivalent systems. *Probabilistic Engineering Mechanics*, 45:22–30, jul 2016.

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